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**Spencer R. Hall, Alan J. Tessier, Meghan A. Duffy, Marianne Huebner, and Carla E. Cáceres. 2006. Warmer does not have to mean sicker: temperature and predators can jointly drive timing of epidemics. *Ecology* 87:1684–1695.**

Appendix A. Equilibria and stability of the  $S-I-Z$  model with predators.

In this appendix, we describe and detail the stability of the equilibria produced by model of susceptible host ( $S$ ), infected host ( $I$ ), and free-floating spore ( $Z$ ) dynamics with predators ( $P$ ). This  $S-I-Z$  model with  $P$  (Eq. 1) produces a single interior equilibrium (i.e.,  $S_i^* > 0$ ,  $I^* > 0$ ,  $Z^* > 0$ ) defined by:

$$S_i^* = \left( \frac{m}{\sigma(d+v)} \right) \left( \frac{d+v+\theta P}{\beta} \right) \quad (\text{A.1a})$$

$$I^* = \sqrt{ \frac{ \frac{1}{2} \left( \frac{b_i - d - v - \theta P}{b_i c} - \left( 1 + \frac{b}{b_i} \right) S_i^* \right) + S_i^* \left( \frac{b}{b_i} (S_i^* - S_b^*) + \frac{1}{2} \left( S_i^* \left( 1 + \frac{b}{b_i} + \frac{\sigma\beta(d+v)}{b_i c m} \right) + \frac{1}{c} \right)^2 \right) }{2} } \quad (\text{A.1b})$$

$$Z^* = \left( \frac{\sigma}{m} \right) (d+v) I^* \quad (\text{A.1c})$$

where  $S_b^*$  is the boundary equilibrium occurring in the absence of the parasite (see Eq. 2a). The equilibril density of  $S_i$  is the ratio of rates of losses to gains of spores times the ratio of rates of losses and gains of infected hosts. Density of infected hosts at  $I^*$  is a complex expression of  $S_b^*$  and  $S_i^*$  and increases with as density-dependence relaxes (i.e., with  $c^{-1}$ ). Finally, equilibril density of spores ( $Z^*$ ) equals the ratio of the rate of spores produced by dead  $I$  divided by the loss rate of spores. Positive density of  $S_b$  and  $I$  requires that predator density not exceed, respectively:

$$P = \frac{b-d}{f} \quad (\text{A.2a})$$

$$P = \left( \frac{d+v}{f} \right) \left( \frac{\sigma\beta(b-d) - b c m}{b c m \theta + \sigma\beta(d+v)} \right) \quad (\text{A.2b})$$

where the latter condition (Eq. A.2b) is, not surprisingly, a more restrictive requirement than the former (Eq. A.2a).

Following standard protocol for stability analysis of ordinary differential equations, we first construct a Jacobian matrix for the  $S-I-Z$  system with predators:

$$\begin{bmatrix} b(1-c(I+2S))-b_I cI-d-fP-\beta Z & -bcS+b_I(1-c(2I+S)) & -\beta S \\ \beta Z & -d-v-\theta fPI & \beta S \\ 0 & \sigma(d+v) & -m \end{bmatrix} \quad (\text{A.3})$$

The Routh-Hurwitz criteria for stability of such a system require that the coefficients of its characteristic polynomial ( $\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3$ ) meet:  $A_1 > 0$ ,  $A_3 > 0$ , and  $A_1 A_2 > A_3$ , where:

$$A_1 = -J_{11} - J_{22} - J_{33} \quad (\text{A.4a})$$

$$A_2 = J_{11}(J_{22} + J_{33}) - J_{12}J_{21} - J_{23}J_{32} + J_{22}J_{33} \quad (\text{A.4b})$$

$$A_3 = J_{21}(J_{12}J_{33} - J_{13}J_{32}) + J_{11}(J_{23}J_{32} - J_{22}J_{33}) \quad (\text{A.4c})$$

and  $J_{n,k}$  are the elements of the Jacobian matrix (Eq. A.3). With some algebra, one can show that  $J_{11} < 0$  always (because  $J_{11} > 0$  at a  $P$  which is too large to fall along the nullcline created by setting  $dS/dt$  to zero). Therefore,  $A_1 > 0$  is always met. Condition  $A_3 > 0$  is always met as long as  $I^* > 0$  (see Eq. A.2b). The third condition defies easy interpretation algebraically. Suffice it to say here that, for the range of parameters considered in this paper, this condition is always met. Thus, the dynamics of the  $S-I-Z$  system behave stably in the examples here.

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